Some Non-Born Phase Shifts*

HENRY BRYSK
Physics Department, Adelphi University, Garden City, New York
(Received 27 June 1963; revised manuscript received 5 November 1963)

Over the past few years, there have been a number of independent attempts to improve on the Born approximation calculation of scattering phase shifts. The similarity of several of the first-order results for central potentials has led to misconceptions as to their interrelationship. The interpretation of the approximations has now been put on a firm basis (sometimes at variance with the previous literature). The alternatives in going to higher order are sorted out and their meaning is clarified. The extensions to noncentral potentials are given. In addition, a numerical exploration of the precision of different higher order results has been carried out.

I. INTRODUCTION

THE solution of the wave equation for scattering by a potential (or its partial-wave decomposition) has most frequently been attempted by the Born approximation series: Set the potential $U=\lambda U_0$, where λ is the strength parameter. Expand Ψ in a power series in λ. Then, in the spirit of perturbation theory, equate the coefficients of each power of λ. It has been known for a long time that the series will diverge if λ is too large, and much effort has gone into investigating the convergence conditions. The most sophisticated analysis[1,2] invokes the Fredholm determinant solution of the integral equation. The determinantal solution takes the form of a ratio of two power series in λ, valid for all regular λ. Apart from minor refinements, the first zero of the Fredholm denominator (viewed as a function of λ) determines the radius of convergence of the Born series. (For attractive potentials, it also locates the first resonance.)

The implication that, in going beyond the first Born approximation, the introduction of a new term in the denominator might be at least as important as proceeding to a higher term in the Born series was not acted on explicitly until recently. There are now several distinct derivations of such an expression, all yielding essentially identical first-order results for a central potential. All have different starting points and independent developments.

It is the purpose of this paper to clarify the connections among the various approaches, trace the similarities and point out the differences. The implications as to the choice among conflicting expansions to higher order are elucidated. While a complete review of the subject has not been attempted, a certain amount of exposition of previously published techniques has been necessary. This occurs primarily where the meaning of the technique has been reinterpreted, particularly as to range of validity and as to expectations on the convergence of iterative procedures (in this respect, some earlier misconceptions are corrected).

Since the aim of an approximation is to attain reasonable numerical values with a limited effort, an extensive computational program has been carried out to ascertain how well the several variants of approximation reproduce exact results when carried through second order (higher for scattering lengths). Results are given explicitly for the $l=0$ phase shifts for square wells and barriers; other cases are alluded to. The quoted numerical work is for a range of parameters for which the Born approximation fails.

As the title states, this paper is concerned with phase shifts, hence with individual terms in a partial-wave decomposition. Such a decomposition is ordinarily useful when only a small number of partial waves contribute significantly to the scattering. In practice, this means in classical terms that the scatterer is smaller than the wavelength or, alternatively, relatively low energy. The question of how accurate a cross section is obtained on summing the phase shifts, which becomes crucial as the energy increases, has been left unanswered because the scope of computation required was impractical for the author. There is no discussion of the quite different techniques that are geared to the high-energy region, such as extensions of the WKB method[3,4] and use of a transformation function.[5]

Over the past few months, several schemes have appeared for transforming the Born series to improve its convergence (or cause it to converge where it did not).[6–8] The motivation is to extend the applicability of perturbation techniques in field theory. While these schemes are usable in principle for potential scattering, the effort required for a specified return is considerably greater than for the non-Born techniques discussed in the present paper. This comment is not intended to depreciate the works in question; it merely points up

* Work supported by the National Science Foundation under Grant NSF-G 23169.

[1] R. Jost and A. Pais, Phys. Rev. **82**, 840 (1951).
[2] W. Kohn, Rev. Mod. Phys. **26**, 292 (1954).
[3] D. S. Saxon, Phys. Rev. **107**, 871 (1957).
[4] S. Rosendorff and S. Tani, Phys. Rev. **128**, 457 (1962).
[5] S. Rosendorff and S. Tani, Phys. Rev. **131**, 396 (1963).
[6] M. Rotenberg, Ann. Phys. (N. Y.) **21**, 579 (1963).
[7] S. Weinberg, Phys. Rev. **130**, 776; **131**, 440 (1963).
[8] M. Wellner, Phys. Rev. **132**, 1848 (1963).

that, while potential scattering has been a convenient proving ground for field theory, the optimum strategy need not be the same in both areas.

II. CENTRAL POTENTIALS

A. Exact Formulas

For a central potential, the partial wave satisfies the integral equation

$$\Psi_l(r)=i^l j_l(kr)-ik\int_0^\infty r'^2dr'U(r') \times\Psi_l(r')j_l(kr_<)h_l(kr_>). \quad (1)$$

(The notation, reasonably standard, is as given by Brysk,[9] except that the partial-wave function is used here rather than just its scattered part; the superscript (1) has been omitted from the spherical Hankel function of the first kind to avoid cumbersomeness further down; and the potential range is denoted by b.) Expansion of the scattering amplitude

$$f(\theta)=\sum_l (2l+1)A_lP_l(\cos\theta) \quad (2)$$

yields the partial-wave amplitudes

$$A_l=-(-i)^l k\int_0^\infty r'^2dr'U(r')\Psi_l(r')j_l(kr'). \quad (3)$$

Solution of the scattering problem consists of solving the Fredholm equation [Eq. (1)] with its kernel whose form changes at $r'=r$, the scattering amplitude being obtained by going to the asymptotic limit of $\Psi_l(r)$ or (in effect iterating an approximate wave function) by substitution into Eq. (3). The fact that $\Psi_l(r)$ and the equations are complex necessitates special precautions in any approximate scheme to preserve the unitarity of the S matrix. The difficulty is commonly resolved by resorting to the choice of differently normalized functions

$$\phi_l(r)=i^{-l}\exp(-i\delta_l)\Psi_l(r), \quad (4)$$

which turns Eqs. (1) and (3) into

$$\sin\delta_l=-k\int_0^\infty r'^2dr'U(r')\phi_l(r')j_l(kr'), \quad (5)$$

$$\phi_l(r)=\cos\delta_l j_l(kr)+k\int_0^\infty r'^2dr'U(r') \times\phi_l(r')j_l(kr_<)y_l(kr_>). \quad (6)$$

This makes everything real, and thereby ensures that unitarity will be preserved in the further calculations.

This familiar set of equations has been elegantly modified by Drukarev[10] for the case of zero angular momentum. His transformation is readily generalized to arbitrary l: When the change in the kernel at $r'=r$ is put in explicitly, Eq. (1) can be rewritten as

$$\Psi_l(r)=j_l(kr)\left[i^l-ik\int_0^\infty r'^2dr'U(r')\Psi_l(r')h_l(kr')\right] +ik\int_0^r r'^2dr'U(r')\Psi_l(r')[j_l(kr)h_l(kr') -h_l(kr)j_l(kr')]. \quad (7)$$

The coefficient of the Bessel function in the first term is a pure number. If a new function $u_l(r)$ is defined by factoring this coefficient out of the wave function (and the Hankel function in the second term is split into a Bessel function and a Neumann function), $u_l(r)$ satisfies a Volterra equation involving only real quantities

$$u_l(r)=j_l(kr)-k\int_0^r r'^2dr'U(r')u_l(r')[j_l(kr)y_l(kr') -y_l(kr)j_l(kr')]. \quad (8)$$

In terms of $u_l(r)$, Eq. (3) becomes

$$A_l=-k\int_0^\infty r'^2dr'U(r')j_l(kr')u_l(r')\Big/ \times\left[1+ik\int_0^\infty r'^2dr'U(r')h_l(kr')u_l(r')\right] \quad (9)$$

and (more attractively), the phase shift is given by

$$\tan\delta_l=-k\int_0^\infty r'^2dr'U(r')j_l(kr')u_l(r')\Big/ \times\left[1-k\int_0^\infty r'^2dr'U(r')y_l(kr')u_l(r')\right]. \quad (10)$$

The new system [Eqs. (8) and (10)] has an obvious superiority over the old [Eqs. (1) and (3), or (5) and (6)] for exact numerical solution. It also turns out to be a very fruitful starting point for the analysis of approximation techniques.

B. Perturbation Expansions

The Born approximation amounts in essence to a perturbation expansion based on Eq. (1), i.e., to setting

$$U=\lambda U_0, \quad \Psi_l=\sum_n \lambda^n\Psi_l^{(n)}, \quad A_l=\sum_n \lambda^n A_l^{(n)} \quad (11)$$

so that

$$A_l^{(0)}=-k\int_0^\infty r'^2dr'U(r')j_l^2(kr'), \quad \text{etc.}\cdots. \quad (12)$$

The Fredholm determinantal solution, on the other hand, leads to a ratio of power series in λ. Using the

[9] H. Brysk, Phys. Rev. **126**, 1589 (1962).
[10] G. F. Drukarev, Zh. Eksperim. i Teor. Fiz. **25**, 139 (1953).

Schwinger formalism, Falk[11] has explicitly evaluated this solution to terms of first order in λ, obtaining

$$\tan\delta_l^{(1)} = -k\int_0^\infty r'^2 dr' U(r') j_l^2(kr') \Big/ \left[1-k\int_0^\infty r'^2 dr' U(r') j_l(kr') y_l(kr')\right]. \quad (13)$$

A perturbation expansion for $u_l(r)$ has been carried by Drukarev,[12] leading to a ratio of power series in λ upon substitution into Eq. (10). To lowest order, Eq. (13) results. It has been proven by Brysk[13] that the Drukarev expansion is identical with the Fredholm solution. Thus, an iterative (successive approximation) solution of the Volterra equation [Eq. (8)] generates the Fredholm determinants, just as an iterative solution of the partial-wave equation [Eq. (1)] generates the Born series. In other words, the effort normally expended on the Born approximation suffices to obtain the Fredholm solution with the Drukarev transformation.

C. Other Approximations

So far, Eq. (13) has been obtained in a weak-potential approximation (i.e., by discarding in the expansion all terms involving the potential to a power greater than one). It also emerges under other circumstances, not necessarily amenable to a perturbation expansion. For instance, Eq. (13) yields the exact solution for a potential $U(r)=U_0\delta(r-a)$, as is readily verified by solving Eq. (8) with the δ function and substituting the answer into Eq. (10).

Equation (13) has been obtained by Brysk[9] from a different approximation. The starting point is Eq. (1), which is rewritten as the sum of terms involving an infinite and an indefinite integral much like Eq. (7) except that, instead of splitting off an integral from 0 to r, an integral from r to ∞ is now split off. The approximation consists of discarding the indefinite integral (the "tail"). This is in the nature of a short-range approximation. It can be viewed equivalently as an approximation of the Green's function in Eq. (1):

$$-ikj_l(kr_<)h_l(kr_>) \approx -ikj_l(kr')h_l(kr). \quad (14)$$

The remaining expression is a linear combination of a Bessel function and a Hankel function (since the infinite integral is a number). It can be written as

$$\Psi_l(r) = i^l j_l(kr) + i^{l+1} A_l h_l(kr), \quad (15)$$

where A_l is the partial wave amplitude as defined by Eq. (2). The value of A_l is obtained by substituting Eq. (15) into Eq. (3). Again Eq. (13) emerges.

The relation to the Drukarev formalism is most easily seen by inserting the approximation into Eq. (8) by raising the limit on the integral in the latter to infinity. Using the exact Eq. (10), this can be recast as

$$u_l(r) = \left[1-k\int_0^\infty r'^2 dr' U(r') u_l(r') y_l(kr')\right] \times [j_l(kr) - \tan\delta_l y_l(kr)]. \quad (16)$$

If Eq. (16) is substituted into the numerator of Eq. (10) and the resultant equation is solved for $\tan\delta_l$, Eq. (13) results again. Thus, while the same phase shift is obtained as in the lowest order of the Drukarev expansion, it is obtained with a different wave function.

If the potential vanishes for $r>a$, then Eq. (15) gives the exact wave function for the region $r>a$ (if A_l is exact). The approximation thus amounts to using the external wave function everywhere when computing A_l from Eq. (3), or in effect performing an iteration on the wave equation with the external wave function as trial function. Swan[14] starts nearly from the last point, although with a different outlook. He uses the alternative normalization of the partial wave functions given by Eq. (4). The essential features of the internal wave function (Swan argues) are that it should join smoothly to the external one and that it should behave as r^l near the origin. On the other hand, if Eq. (15) were extended to the origin, the first term would behave as r^l but the second one as r^{-l-1}. In principle then, it should be possible to express the wave function everywhere by

$$\phi_l(r) = \cos\delta_l j_l(kr) - \sin\delta_l g_l(r) y_l(kr), \quad (17)$$

where $g_l(r)$ is a smooth function which becomes 1 at the cutoff (or more generally goes to 1 as $r\to\infty$) and which behaves as r^{2l+1} as $r\to 0$. [It should be noted that this is *not* equivalent to multiplying the second term in Eq. (15) by a similar factor]. Substitution of Eq. (17) into Eq. (5) leads to

$$\tan\delta_l = -k\int_0^\infty r'^2 dr' U(r') j_l^2(kr') \Big/ \times\left[1-k\int_0^\infty r'^2 dr' U(r') g_l(r') j_l(kr') y_l(kr')\right], \quad (18)$$

which differs from Eq. (13) only in the factor $g_l(r')$ in the denominator. The essence of the Swan approximation consists of guessing at the unknown function $g_l(r)$ subject to the general conditions just enumerated. The primary choice is

$$g_l(r) = \tfrac{1}{2}[(2l+3)(r/R)^{2l+1} - (2l+1)(r/R)^{2l+3}], \quad r\le R \quad (19)$$
$$= 1, \quad r>R$$

[11] D. S. Falk, Phys. Rev. **129**, 2340 (1963).
[12] G. F. Drukarev, Vestnik Leningrad Univ. **22**, 65 (1958).
[13] H. Brysk, J. Math. Phys. **4**, 1536 (1963).
[14] P. Swan, Nucl. Phys. **18**, 245 (1960).

where R is the cutoff of the potential. There is usually some arbitrariness in the choice of precisely where to truncate the potential, and Swan views this as supplying an adjustable parameter to be exploited to fit an additional bit of information. His own preference is to match the scattering length

$$a=-\lim_{k\to 0} k^{-1}\tan\delta_0, \tag{20}$$

a quantity which is easier to obtain accurately (for instance, it can be bounded by a variational principle[15]). For longer tailed potentials, Swan resorts to more elaborate form factors.[16,17]

The formally sound procedure for achieving a systematic improvement in approximation lies in carrying out the Fredholm determinantal solution (or, equivalently and more simply, iterating the Drukarev expansion) since ultimately the expansion will converge to the exact answer. It does not follow *a priori* that this is the optimum procedure when one stops in low order (especially for strong potentials), as formal convergence does not imply anything as to the early rate of convergence. A valid alternative to the second-order Fredholm result might be a variational calculation using as trial function the Born wave function[18] or (more elaborately) the Brysk or Swan wave functions. Another plausible course is outlined by Swan[19]: If Eq. (6) is schematized by the operator relation

$$\phi_l=f_l+G_l\phi_l, \tag{21}$$

continued iteration yields

$$\phi_l=f_l+G_l\phi_l=f_l+G_lf_l+G_l^2\phi_l=f_l+G_lf_l+G_l^2f_l+G_l^3\phi_l=\cdots. \tag{22}$$

Neglect of the last term on the right leads to the Born series. The Swan iteration consists of substituting for ϕ_l on both sides of Eq. (22) the Swan wave function [Eq. (17)] and solving for the parameter $\tan\delta_l$. If the last term on the right ultimately dwindles, the Born series and the Swan iteration will approach the same limit. On this basis, Swan suggests that he has obtained in effect a reordering of the Born series with an improved early rate of convergence. There are no formal results on convergence beyond this. In some particular instances, the Swan iteration appears to converge to the exact answer while the Born series oscillates; thus the two remain both numerically and conceptually apart. In other cases, both oscillate.

D. Numerical Results

In this section the focus shifts from formal relations to numbers. The purpose is to ascertain by example just how effective the approximation technique is: How dependable are the first- and second-order results? What is the optimum strategy beyond first order, which second order approach is preferable? How much is gained by going to a moderately higher order? How high an order does it take to get an accurate answer? In order for the numerical exploration to be instructive, it is necessary to push beyond the comfortable range of parameters to borderline cases—to find out where the borderline is. The numerical results quoted are for parameters for which the Born approximation fails completely and the more elaborate techniques are overextended; this should not obscure the successes under less demanding conditions.

The first set of computations is for the scattering lengths in neutron-proton scattering as obtained by the various iterative schemes up to terms of fourth order. The scattering length is a rather sensitive function and the potential is of fair strength. A comparison is thus obtained of the rate of convergence of the alternative approaches when the first order is not adequate.

The second set of computations is of $l=0$ phase shifts for square wells and barriers, carried through second order in the different versions. Here the emphasis is on discovering the dependence of the accuracy on the range and on the magnitude of the potential when the calculation is restricted to first or second order, and also on comparing the performance of the second-order variants. In essence, the questions met are: When can a reasonable answer be obtained with moderate effort? What is the best way to do it?

(1) *Scattering Lengths*

The scattering length has intrinsic physical interest as characterization of threshold behavior. For present purposes it is a rather convenient quantity because in the low-k limit the approximation integrals for the simplest potentials become elementary (though collectively tedious) so that it is possible to push on to higher orders analytically. It should be noted that the scattering length does not represent a weak-potential limit: while kb tends to zero, U_0/k^2 tends to infinity, and the product U_0b^2 is kept fixed. Swan's set of neutron-proton potentials are quite suitable for a test of approximations. The potential strength is moderate, too great for the Born approximation yet weak enough that iterative schemes have a chance of success; and the scattering length is a very sensitive quantity, as is evident from its change of sign as well as magnitude for the not very drastic difference in well parameters between the singlet and triplet states.

For the square-well potentials, Swan has computed the scattering lengths to third order in the Born series and in his own.[19] The corresponding Fredholm determinantal solution (Drukarev expansion) has now been carried out. In addition, the Swan wave function has

[15] L. Spruch and L. Rosenberg, Phys. Rev. **116**, 1034 (1959).
[16] P. Swan, Nucl. Phys. **21**, 233 (1960).
[17] P. Swan, Australian J. Phys. **16**, 177 (1963).
[18] J. Sokoloff and M. Hamermesh, Ann. Phys. (N. Y.) **2**, 157 (1957).
[19] P. Swan, Nucl. Phys. **27**, 620 (1961).

TABLE I. Scattering length (in F) for a square well of radius 2.0048F and depth $0.87704F^{-2}$ (Swan's neutron-proton triplet state). Entries in the second row are successive approximations obtained from the Born series, in the third row from the Swan series, in the fourth row from the Fredholm determinantal solution, in the fifth and sixth rows from Schwinger variational calculations based on the Swan and Brysk wave functions.

	$a^{(1)}$	$a^{(2)}$	$a^{(3)}$	$a^{(4)}$
Exact	5.380	5.380	5.380	5.380
Born	−2.355	−5.676	−10.416	−17.185
Swan	5.745	5.611	5.559	5.534
Fredholm	3.089	6.231	5.333	5.378
var. Swan		5.377		
var. Brysk		5.864		

been used in a Schwinger variational calculation. The results are given for the triplet state in Table I. The pattern is similar for the singlet state. The shortcomings of the Born series in this case need no further comment. The Swan approximation starts out fairly close, and converges to the exact answer in what might be described as an "overdamped" manner. The Fredholm solution is poor in first order, but ultimately overtakes the Swan series in convergence. The variational results are remarkably good; the agreement is even more striking when the calculation is extended to terms of order k^2: The correct effective range is obtained to three significant figures for both states. This demonstrates that Swan's choice for the form of the wave function is very apt in the particular instance. In fact, the point of the variational calculation was to explore this question. For the exclusive purpose of determining accurately the scattering length, the same computational effort would have been more profitable expended on a multiparameter Spruch-Rosenberg[15] calculation, with the overwhelming advantage of known dependability. (To complete the comparison of wave functions, the scattering length obtained from a variational calculation based on the unperturbed wave function is exactly the same as the first-order Swan result, and a variational calculation based on the Brysk wave function gives closely similar answers. Incidentally, Weinberg's prescription for the scattering length,[7] which requires about the same effort as a variational calculation, also yields the first-order Swan result.)

While the square-well potential is the tidiest for computation, its sharp cutoff makes it atypical. A simple continuous potential with a tail is provided by the exponential. The above set of scattering length calculations have been repeated for exponential wells with Swan's parameters. To save space, the full numerical results are not quoted. Again, the Fredholm solution starts off poorly but zeros in quickly beyond the second order, though not as fast as for the square well (the average error is 7% in fourth order as against 0.1% before); the poorer convergence is presumably chargeable to the tail. Calculations with Swan's polynomial form factor [Eq. (19)] show an extreme sensitivity to the choice of cutoff radius (affecting even the sign of the first-order singlet scattering length). Convergence of the Swan iteration is poor when the first-order result is not close; the variational calculation does less well than the second-order result.

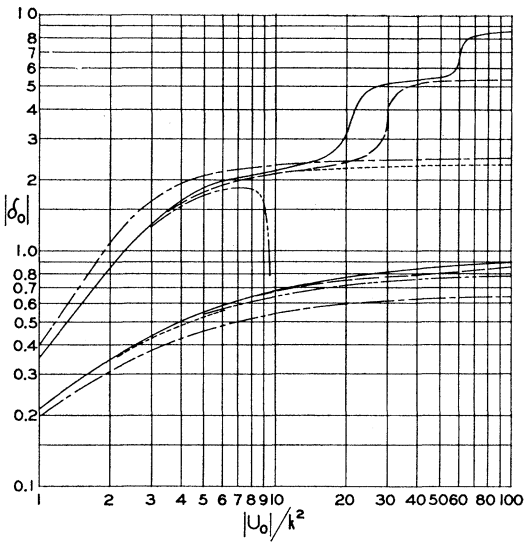

FIG. 1. S-wave phase shifts for square wells (upper set of curves) and barriers (lower set of curves) with range $kb=1$ and varying potential depths or heights. The curves represent the exact solution (solid line), the first-order Swan approximation (·····), the Fredholm determinantal solution in first order (—·—·) and second order (—··—··), and a Schwinger variational calculation based on the Brysk wave function (— — —). (From considerations of clarity of display, the second-order Fredholm is not shown past its discontinuous jump.)

(2) *S-Wave Phase Shifts for Square Wells and Barriers*

For a set of neutron-proton scattering potentials adjusted to fit the experimental scattering length and effective range for the singlet and triplet states (corresponding to a radius of about 2 F and a well depth of about $1/2F^{-2}$), Swan[14,16] has computed both exact values of the phase shifts and his approximation to them for energies up to some 200 MeV ($kb \lesssim 4$) for square, Gaussian, exponential, and Yukawa wells for δ_0 and as many other phase shifts as are competitive (up to δ_6). For the s wave the results are good to 2–3% on the average, a remarkable success. Without the form factor, the errors are several times larger, and comparable results can only be obtained in higher order. Judging by the rapid deterioration in variational calculations based on the Swan wave functions as k increases, the phase shifts are much better than the wave functions.

From the more general point of view of studying scattering approximations, it is obviously desirable to examine a much wider selection of potential radii and depths (or heights). In the natural set of dimensionless

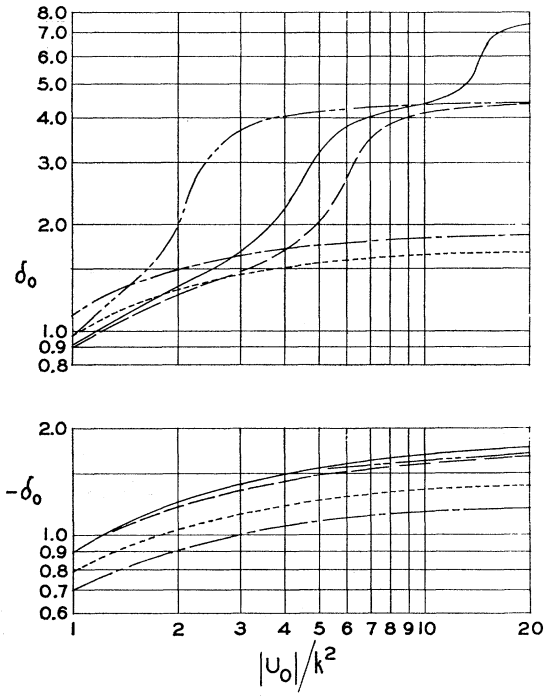

FIG. 2. Same as Fig. 1 except for $kb=2$.

units, this means mapping out the phase shifts as kb and U_0/k^2 are varied (with U_0 either positive or negative). For a full survey of resources through second order, exact $l=0$ phase shifts have been compared with the first-order Fredholm (or Drukarev or Brysk) and Swan solutions, the second-order Fredholm solution, second-order Swan-type iterations on Brysk and Swan, and Schwinger variational calculations with the unperturbed wave function and with the Brysk and Swan wave functions. The second-order Swan is useful in improving the already good first order for very small kb, but not remarkable otherwise. Generally, the Swan iterations show no consistent improvement over the first order for stronger potentials. The same comment applies to the variational calculation based on the Swan wave function. The variational calculation with the unperturbed wave function is numerically close to the first-order Swan for smaller kb; it is a bit better for positive potentials, a bit worse for negative. The remaining variants are exhibited for $kb=1$, 2, and π as a function of the potential in Figs. 1–3. Brysk[9] has graphed his first-order results for $kb=0.1$, 0.5, 1.0, and 2.0. For $kb=0.1$ and 0.5, these are already quite good; on the scale of the curves, the deviation of the Swan and higher order results from the exact would barely be noticeable (except for the resonance at the upper end of the 0.5 curve).

Even at a superficial glance, the rapid weakening of calculational resources as kb increases is striking. This is the more damaging in that greater precision is required just then because the phase shifts are larger and different partial waves interfere. The effect is expected for the Swan approximation, which is essentially a small kb rather than a weak-potential approximation, but it is stronger than anticipated on formal grounds for the Fredholm solution to low order.

The simplest approximation considered is the first-order Fredholm (or Drukarev or Brysk). The effort required is about twice what the Born approximation takes (two comparable integrals instead of one) and the precision is overwhelmingly superior even for very small phase shifts (say, 0.01), while the range of applicability is vastly greater. In the range of parameters of this section, this is the approximation to use if 10 to 30% quality results are satisfactory (getting worse with larger kb), short of the resonances.

Next in complexity is the Swan approximation. An exact determination of the scattering length is normally required, and the form factor renders the computations more tedious. Short of resonances, this approximation is good to a few percent for small kb regardless of potential strength. As kb increases, the form factor loses its effectiveness. In fact, for large kb

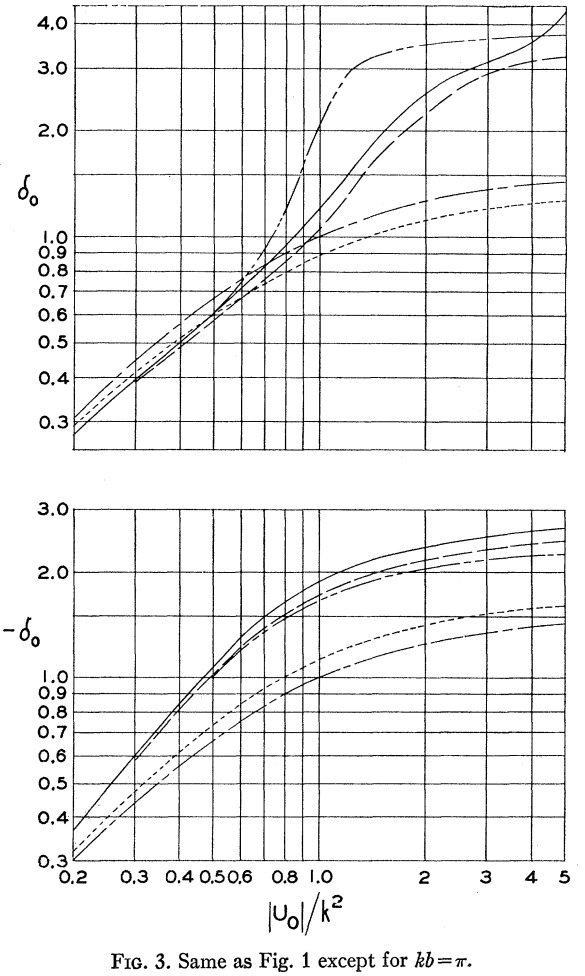

FIG. 3. Same as Fig. 1 except for $kb=\pi$.

the variational calculations would seem to indicate that the shape of the wave functions is better without it. The Swan approximation does better for negative than positive potentials of the same strength (again short of resonances). Even for $kb=\pi$, it still holds up for the neutron-proton potential (about $U_0=-0.5k^2$) to within 1%, as against 30% for a comparable positive potential, but fails rapidly as the potential strength is increased. For small kb, the Swan approximation can be viewed as a very sophisticated interpolation formula: For small U_0, it yields the correct Born limit. For large U_0, adjustment to the exact scattering length via the form factor amounts to fixing the asymptotic limit correctly. A reasonably shaped wave function links the two limits.

The second-order Fredholm (or Drukarev) solution is more complicated in that it involves double integrals. (On the other hand, as against the Swan approximation, it does not require the exact evaluation of the scattering length nor carry along the form factor.) For the larger values of kb, it is markedly better than the Swan approximation for positive potentials. (For $kb=2$ it remains within an 8% error margin, four times better than Swan. For $kb=\pi$, it survives to a significantly higher U_0.) For negative potentials, it does give one resonance, but for appreciably too shallow a well (for smaller kb, there is an abrupt drop in the phase shift followed by a discontinuous jump); beyond that point, the mistiming is too large for dependable phase shifts to be obtained, though at least there is a warning of the approach of the true resonance.

Most consistently successful, but also more work, is the Schwinger variational calculation based on the Brysk wave function. Through $kb=2$, it tends to be just barely better for positive potentials and for negative potentials prior to the first resonance. It is clearly superior for the first resonance, which it reproduces with a lag but much more nearly correct than the second-order Fredholm solution. Higher resonances are ignored. The most clear-cut success is for $kb=\pi$ where phase shifts for positive potentials and for negative potentials through the first resonance are given with reasonable accuracy while all other approximations fail at very low U_0.

III. NONCENTRAL POTENTIALS

The exact spherical harmonic decomposition for an arbitrary noncentral potential is derived by Brysk.[9] With the wave function and the potential both expanded in terms of spherical harmonics, the partial waves are coupled:

$$\Psi_{lm}(r)=\delta_{m0}i^l j_l(kr)-ik\sum_{nps}(-)^{m-s}(2p+1)\times C(pln;00)C(pln;m-s,-m)\times\int_0^\infty r'^2dr' j_l(kr_<)h_l(kr_>)\Psi_{ns}(r')U_{pm-s}(r'), \quad (23)$$

where the C's are Clebsch-Gordan coefficients. Brysk applies his Green's function approximation [Eq. (14)] to obtain for the scattering coefficients A_{lm} the set of coupled linear equations

$$A_{lm}=-k\sum_{np}i^{n-l}(2p+1)C(pln;00)\times\Big[C(pln;m,-m)\int_0^\infty r^2dr j_l(kr)j_n(kr)U_{pm}(r)+i\sum_s A_{ns}C(pln;m-s,-m)\times\int_0^\infty r^2dr j_l(kr)h_n(kr)U_{pm-s}(r)\Big]. \quad (24)$$

A generalization of the Drukarev transformation for noncentral potentials is possible, though not as simple as for central potentials. Just as with Eq. (1), Eq. (23) can be rewritten as the Bessel function times a numerical coefficient N_{lm} which includes infinite integrals, plus an integral from 0 to r. Because of the coupling of the partial waves, however, a simple factorization no longer works. Taking a cue from Drukarev's treatment of electron-atom collisions,[10] the transformation is achieved by setting

$$\Psi_{lm}(r)=\sum_{\alpha\beta}N_{\alpha\beta}u_{lm}{}^{\alpha\beta}(r). \quad (25)$$

This substitution turns both sides of Eq. (23) into sums over α and β every term of which contains $N_{\alpha\beta}$ as a factor. A solution of the equation can be obtained by requiring that the cofactor of each $N_{\alpha\beta}$ should vanish, leading to the system of coupled Volterra equations

$$u_{lm}{}^{\alpha\beta}(r)=\delta_{\alpha l}\delta_{\beta m}j_l(kr)-k\sum_{nps}(-)^{m-s}(2p+1)\times C(pln;00)C(pln;m-s,-m)\times\int_0^r r'^2dr' u_{ns}{}^{\alpha\beta}(r')U_{pm-s}(r')\times[j_l(kr)y_l(kr')-y_l(kr)j_l(kr')]. \quad (26)$$

Once the Volterra equations are solved and the infinite integrals involving the $u_{lm}{}^{\alpha\beta}(r)$'s are carried out, there remains a system of linear algebraic equations for the $N_{\alpha\beta}$'s.

A perturbation expansion can be applied to the extended Drukarev transformation, at least in principle. To lowest order, only the first term on the right-hand side of Eq. (26) is retained. There results an explicit system of linear algebraic equations

$$N_{lm}=\delta_{m0}i^l-ik\sum_{nps}(-)^{m-s}(2p+1)\times C(pln;00)C(pln;m-s,-m)N_{ns}\times\int_0^\infty r^2dr h_l(kr)j_n(kr)U_{pm-s}(r). \quad (27)$$

The scattering coefficients are now obtained by sub-

stituting the solution to Eq. (27) into

$$A_{lm}=-(-i)^l k \sum_{nps}(-)^s(2p+1) \times C(pnl;00)C(pnl;m-s,-m)N_{ns} \times \int_0^\infty r^2 dr j_l(kr) j_n(kr) U_{pm-s}(r). \quad (28)$$

For a noncentral potential, Brysk's answer from the Green's function approximation [Eq. (24)] and the Drukarev expansion result [Eqs. (27) and (28)] differ even in lowest order. The Fredholm solution consists in forming the determinantal integrals without first expanding the potential into spherical harmonics[1] and is clearly distinct from either of the above. Thus, the mutual agreement is limited to central potentials, and there is no underlying equivalence: Different ways of thought lead to different results in general, though there may be coincidence in a restricted range. Numerical comparisons would be interesting, but a meaningful analysis would require a sizeable digital computer effort (which is not feasible for the author).

ACKNOWLEDGMENTS

The author found discussions with S. T. Epstein and L. Spruch and correspondence with P. Swan very helpful. He is indebted to G. F. Drukarev for acquainting him with the latter's work.